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Range Maximization of a Surface-to-Surface Missile with In-Flight Inequality Constraints

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The use of aerodynamic lift during (planar) re-entry is considered in maximizing the range of a surface-to-surface missile. The attitude history (which also gives thrust direction) from launch through re-entry is optimized by the steepest-ascent procedure to maximize the surface range at impact. Two inequality constraints are included: 1) a limit on the pitch rate during boost; and 2) a limit on the normal load during re-entry. The results show that a substantially different maximum-range trajectory is obtained with modulated lift rather than with purely ballistic re-entry. The range increase for a 2300-naut-mile ballistic missile is significant if the normal load limit is over 5 g's.

Introduction

THE optimization of rocket vehicle trajectories has been and continues to be of active interest. One major division considers the (vertically launched) surface-to-surface missile. Probably the most common measure of performance is the range capability with a given payload or, conversely, the payload achievable for a specified range. For ballistic re-entry vehicles this problem is one of ascent trajectory optimization. Over a two-dimensional nonrotating spherical earth, the trajectory is entirely determined from the vehicle's (pitch) attitude history during boost. This case has received ample investigation. Generally speaking, the optimal ascent consists of an initial pitch-over period followed by an essentially gravity turn. The particular optimization depends largely on the model assumed for the pitch-over. The use of aerodynamic lift during re-entry offers two advantages: range can be increased, and some measure of interceptor evasion may be gained through maneuvering. We shall consider only the range maximization problem.

We include two inequality constraints that are felt to be particularly relevant to this problem. The first is that the pitch-rate during boost is limited to a maximum value. The missile flies vertically for a prescribed interval and then begins pitching down. We anticipate that the initial pitch-rate will be the maximum value, but this is not demanded. The second constraint is a limitation on the normal force that may be permitted during re-entry. In this investigation the vehicle at re-entry is the same as it is at booster burnout. A vehicle that was designed to have additional lifting surfaces during re-entry could withstand greater normal loads and thus could achieve a greater range increase.

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In this two-dimensional analysis the lift is determined by the vehicle's attitude (and, of course, the dynamic pressure). Hence, the entire trajectory is determined by the attitude, or the equivalent angle of attack which we shall actually use for convenience, program. The angle of attack gives thrust direction and aerodynamic forces during boost and the aerodynamic forces during the glide. Our goal is to obtain the angle-of-attack history (the control program) which maximizes the impact range while satisfying the inequality constraint relations.

For this problem the necessary conditions for the desired optimal solution are readily available, going back as far as Valentine in 1937.[6] Until recent years, however, these would have been only of academic interest. The currently available digital computers make iterative solution of many variational problems now within reach. Although more than one method has been used successfully in certain problems, the steepest-ascent (or gradient) method has been most useful to date in atmospheric trajectory optimization. This scheme, originated independently by H. J. Kelley and by A. E. Bryson Jr., has become a well-known optimization tool. Relatively little has been published thus far with inequality constraints, and a major purpose of this paper is to present an interesting problem including them.

In the steepest-ascent technique as described in Refs. 1–3, each successive improvement is the greatest attainable for a selected value of the integral of the control variable change squared multiplied by a time variable weighting function.† The weighting function, which to a large extent determines the efficiency of the iteration procedure, has yet received little attention. In this problem the sensitivity to angle-of-attack changes is many times greater in booster than in re-entry. The steepest-ascent method without the weighting function would have poor convergence properties for the re-entry angle-of-attack program. The weighting function is used to

† The integral involves a quadratic form in all of the control variable changes if there is more than one.

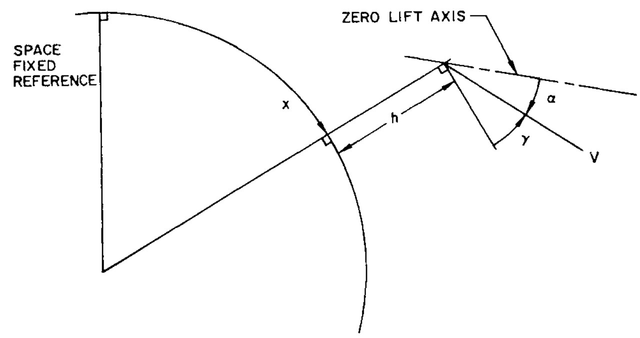

Fig. 1 Nomenclature for analysis of planar motion of a mass center about a nonrotating spherical earth.

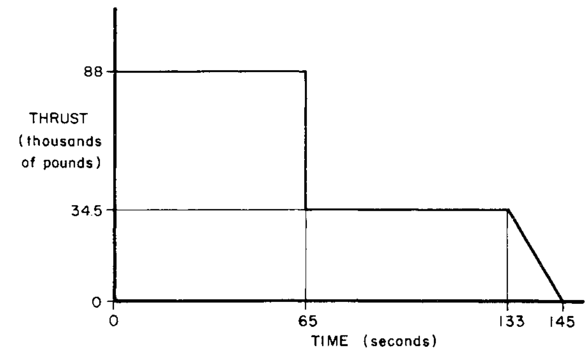

Fig. 2 Thrust history of the two-stage surface-to-surface missile.

produce a suitable balance between the control program changes in boost and in re-entry in the iteration procedure.

Equations of Motion

In this analysis we assume powered, planar motion over a spherical, nonrotating earth. The nomenclature for the problem is shown in Fig. 1. The equations of motion are

$$\dot{V} = \frac{T}{m}\cos\alpha - \frac{D}{m} - g\sin\gamma \tag{1}$$

$$\dot{\gamma} = \frac{T}{mV}\sin\alpha + \frac{L}{mV} - \left(\frac{g}{V} - \frac{V}{R+h}\right)\cos\gamma \tag{2}$$

$$\dot{x} = \frac{V}{1 + (h/R)}\cos\gamma \tag{3}$$

$$\dot{h} = V\sin\gamma \tag{4}$$

where

$\dot{(\ )}$ = $(d/dt)(\ )$
V = velocity magnitude
γ = flight path angle relative to the local horizontal
x = surface range
h = altitude
m = vehicle mass
g = $g_0[R/(R+h)]^2$, acceleration due to gravity
g_0 = acceleration of gravity at earth's surface
R = radius of the earth
D = $C_D(\rho V^2 S/2)$, drag force
L = $C_L(\rho V^2 S/2)$, lift force
T = $T(t)$, thrust
α = angle of attack
ρ = $\rho(h)$, density of atmosphere (ARDC standard atmosphere used)
S = 16 ft^2, reference area
C_D = $C_D(\alpha, M$, stage), drag coefficient
C_L = $C_L(\alpha, M$, stage), lift coefficient
M = Mach number, V/a
a = $a(h)$, speed of sound

The analysis used vehicle characteristics representative of an IRBM. Two stages were assumed, with thrust and mass histories shown in Figs. 2 and 3, respectively. Lift and drag coefficients are shown as functions of α for selected Mach numbers in Figs. 4 and 5. The larger α values in Fig. 5 are shown because the second-stage coefficients are used for the re-entry vehicle, which reaches these higher values in some cases.

Inflight Inequality Constraints

We assume that the vehicle is launched and flies vertically for 3 sec. At that time it begins pitching down. The pitch

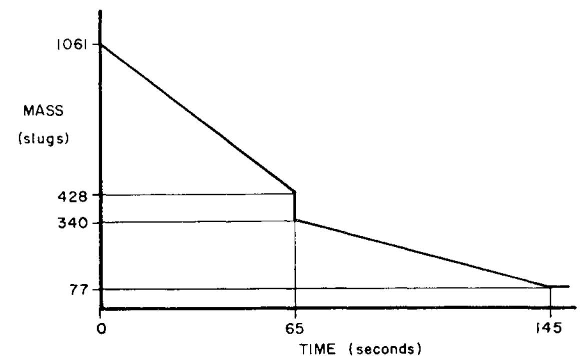

Fig. 3 Mass history of the two-stage surface-to-surface missile.

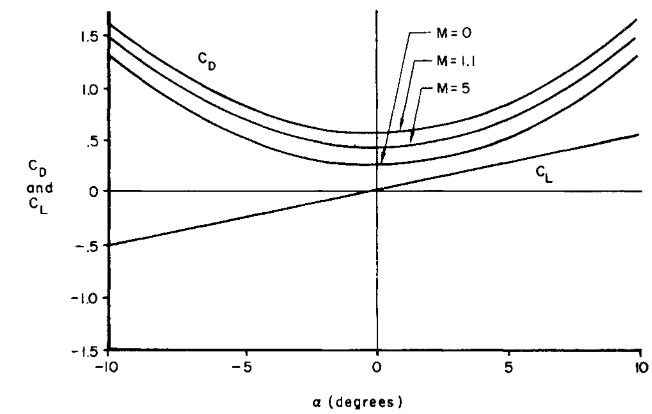

Fig. 4 First-stage aerodynamic coefficients.

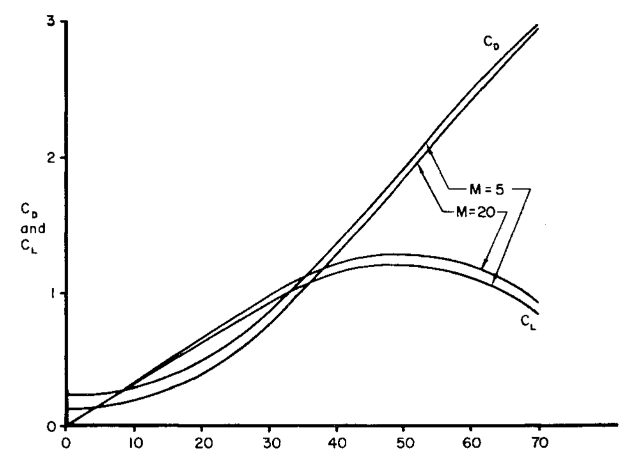

Fig. 5 Second-stage and re-entry aerodynamic coefficients.

rate is limited by a maximum value, chosen as 0.058 rad-sec^{-1}. The pitch rate is $\dot{\theta}$, where

$$\theta = \alpha + \gamma \quad (5)$$

We assume that the maximum $\dot{\theta}$ is used from $t = 3$ sec until t_1 (where t_1 may approach 3 sec). In this interval, then, we use our knowledge of θ to write

$$\alpha = 1.57 - 0.058(t - 3) - \gamma \quad (6)$$

This is the equality relation for the pitch rate inequality constraint boundary.

During re-entry, the normal force-to-weight ratio

$$N = \frac{[L^2 \cos^2\alpha + D^2 \sin^2\alpha]^{1/2}}{mg_0} \quad (7)$$

is constrained to be less than or equal to a prescribed value N_{max}. When $N = N_{max}$, Eq. (7) is used to solve for α. We note that more than one period of $N = N_{max}$ may occur.

Approach to the Problem Solution

We shall not present the details of the steepest-ascent procedure here, as they are available in the literature, for example, in Refs. 1–5. For the most part, the analysis used here is described in Ref. 2.

The problem is to find that $\alpha(t)$ program which will maximize the surface range at impact while satisfying the inequality constraints. The technique is to choose a nominal $\alpha(t)$ program, find a nominal trajectory, and integrate the associated adjoint equations backward from the terminal point in order to make a diagnosis from which to calculate an improved $\alpha(t)$ program. The process is continued until no further useful improvement can be made.

The scheme relies on a linearized perturbation theory. The $\alpha(t)$ program is changed by an amount $\delta\alpha(t)$ which is sufficiently small for the linearization of the equations describing small perturbations about the nominal trajectory to have acceptable validity. "Sufficiently" can only be determined experimentally. (The best method of determination is still a debatable subject.)

The equations of motion are already recorded. The adjoint differential equations, when not on a constraint boundary, are

$$\dot{\lambda}_v - \lambda_v \left(\frac{2D}{mV} + \frac{D}{C_D ma}\frac{\partial C_D}{\partial M}\right) + \lambda_\gamma\left[\frac{L}{mV^2} + \left(\frac{g}{V^2} + \frac{1}{R+h}\right)\cos\gamma + \frac{L}{C_L maV}\frac{\partial C_L}{\partial M} - \frac{T}{mV^2}\sin\alpha\right] + \lambda_x \frac{\cos\gamma}{1 + (h/R)} + \lambda_h \sin\gamma = 0 \quad (8)$$

$$\dot{\lambda}_\gamma - \lambda_v g \cos\gamma + \lambda_\gamma \left(\frac{g}{V} - \frac{V}{R+h}\right)\sin\gamma - \lambda_x \frac{V\sin\gamma}{1 + (h/R)} + \lambda_h V\cos\gamma = 0 \quad (9)$$

$$\dot{\lambda}_x = 0 \quad (10)$$

$$\dot{\lambda}_h - \lambda_v \left(\frac{D}{m\rho}\frac{d\rho}{dh} - \frac{D}{C_D}\frac{M}{ma}\frac{\partial C_D}{\partial M}\frac{da}{dh} - \frac{2g\sin\gamma}{R+h}\right) + \lambda_\gamma\left[\frac{L}{mV\rho}\frac{d\rho}{dh} - \frac{LM}{C_L mVa}\frac{\partial C_L}{\partial M}\frac{da}{dh} + \left(\frac{2g}{V} - \frac{V}{R+h}\right)\frac{\cos\gamma}{R+h}\right] - \lambda_x \frac{V\cos\gamma}{R[1 + (h/R)]^2} = 0 \quad (11)$$

In an interval in which $N = N_{max}$, the adjoint equations are modified to account for the constraint relation. In such an interval, the term

$$\mu(\partial N/\partial V) = \mu(2N/V) \quad (12)$$

must be added to Eq. (8), and the term

$$\mu(\partial N/\partial h) = \mu(N/\rho)(d\rho/dh) \quad (13)$$

must be added to Eq. (10), where

$$\mu = \frac{\lambda_v[(D/mC_D)(\partial C_D/\partial\alpha)] - \lambda_\gamma[(L/mVC_L)(\partial C_L/\partial\alpha)]}{\partial N/\partial\alpha} \quad (14)$$

$$\frac{\partial N}{\partial\alpha} = \frac{(-L^2 + D^2)\sin\alpha\cos\alpha + (L^2/C_L)(\partial C_L/\partial\alpha)\cos^2\alpha + (D^2/C_D)(\partial C_D/\partial\alpha)\sin^2\alpha}{mg_0(L^2\cos^2\alpha + D^2\sin^2\alpha)^{1/2}} \quad (15)$$

There would normally be thrust terms in Eq. (14), but since the re-entry is unpowered these terms have been omitted.

In this problem there are no terminal constraints other than the stopping condition of impact. Thus, only a single solution of the adjoint equations, obtaining the influence functions on the terminal range, is required. The boundary conditions on the adjoint variables are

$$\left.\begin{aligned} \lambda_x(t_f) &= 1 \\ \lambda_h(t_f) &= -(\dot{x}/\dot{h})_{t=t_f} \\ \lambda_\gamma(t_f) &= 0 \\ \lambda_v(t_f) &= 0 \end{aligned}\right\} \quad (16)$$

where the terminal time t_f is determined by

$$\Omega[x(t_f), t_f] = h(t_f) = 0 \quad (17)$$

To begin the successive-improvement process, a nominal $\alpha(t)$ program is chosen (with $\alpha = 0$ for $t < 3$ sec). This selected program is compared to α from Eq. (6). If the nominal α is more negative, α from Eq. (6) is used. At the time t_1 when the two programs become equal, the constraint boundary is left and the selected α program is used for the remainder of the trajectory, except when $N = N_{max}$. In this problem t_1 will always be greater than 3 sec.

The complete trajectory is calculated, and the coefficients for the adjoint equations (8–11), modified by (12) and (13), are stored on tape. The adjoint equations are then integrated backward from the end point to t_1. It is unnecessary to integrate past t_1, because from $t = 0$ to t_1 the α program is constrained. We must, however, make provision for changing t_1 in the improvement process.

Using the steepest-ascent procedure described in Ref. 2, we calculate a change $\delta\alpha(t)$ in the α program for $t \geq t_1$, regions of $N < N_{max}$. In these regions, we have

$$\delta\alpha(t) = W^{-1}(t)\Lambda(t)(dP/I^{1/2}) \quad (18)$$

where

$$\Lambda = \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial\alpha} \qquad \boldsymbol{\lambda} = \begin{bmatrix}\lambda_v \\ \lambda_\gamma \\ \lambda_x \\ \lambda_h\end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix}\dot{V} \\ \dot{\gamma} \\ \dot{x} \\ \dot{h}\end{bmatrix} \quad (19)$$

$(\ )^T$ is the transpose of $(\ )$, $W(t)$ is a weighting function chosen by the optimizer,

$$(dP)^2 = \int [\delta\alpha(t)]^2 W(t)dt \quad (20)$$

is an input number governing the step size in $\delta\alpha$, and

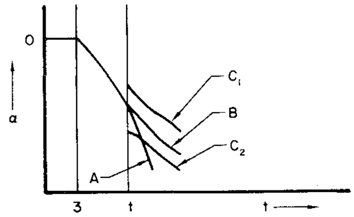

Fig. 6 Representation of the α program in the neighborhood of $t = t_1$.

$$I = \int [\lambda^T(t) \frac{\partial f}{\partial \alpha}(t)]^2 W^{-1}(t)dt \qquad (21)$$

where the integration in (20) and (21) is over all $t > t$, except when $N = N_{max}$. The preceding expressions are considerably simpler than the general ones in Ref. 2, because there are no terminal constraints other than the stopping condition in this analysis. The techniques of determining changes in t_1 and in the beginning and end of the $N = N_{max}$ interval are described in the following discussion.

Consider Fig. 6, where curve A is the beginning of the α program resulting from maximum pitch rate. Curve B represents the α program used in the initial trajectory. Curves C_1 and C_2 are the two possibilities for calculated new $\alpha(t)$ programs, $t \geq t_1$. For negative $\delta\alpha(t_1)$, the natural choice of the next t_1 is the intersection of curves A and C_2. There is no correspondingly easy choice when $\delta\alpha(t_1)$ is positive.

The procedure followed here is to choose the next t_1 as the time at which α of curve B has the same value as the new $\alpha(t_1)$. This α is then used until t reaches the old t_1. For $\delta\alpha(t_1)$ of either sign, then, we have continuity of $\alpha(t)$, and t_1 will move in the desired direction. As the extremal is approached, it may be expected that t_1 will oscillate about its stationary value, but with a continually decreasing amplitude of oscillation, since the gradient to which $\delta\alpha(t_1)$ is proportional is approaching zero at the same time.

For the re-entry portion of the trajectory, the scheme selected is even simpler. In Fig. 7, curve A is the originally selected α program, and curve B is the one used to keep $N \leq N_{max}$. From t_{on} to t_{off}, then, $N = N_{max}$.

Either C_1 or C_2 may represent the new α program for $t \leq t_{on}$, and C_3 or C_4 the new α program for $t \geq t_{off}$. For purposes of choosing a complete nominal α program, the appropriate $\alpha(t_{on})$ is connected by a linear curve to the appropriate $\alpha(t_{off})$. When the program is used, new values of t_{on} and t_{off} will occur, with the α curve having a shape similar to B. In the limit as the optimum is approached, $\delta\alpha$ at t_{on} and t_{off} will be zero as well as everywhere else for $N < N_{max}$ (and $t > t_1$), and some curve like B will be the stationary solution. This particular simple scheme is suitable only when it is clear that the resultant α curve will lie entirely to one side of the linear curve drawn except perhaps near t_{on} or t_{off}. In this case each successive improvement will make an acceptably small change in the $\alpha(t)$ program actually used.

Results and Discussion

There are two trajectories that will serve well as references. One is the purely ballistic re-entry. The other is with neither pitch rate nor normal force limited. Both of these cases can

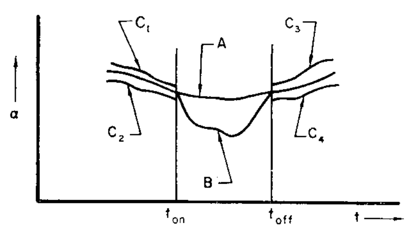

Fig. 7 Representation of the α program for an $N = N_{max}$ interval.

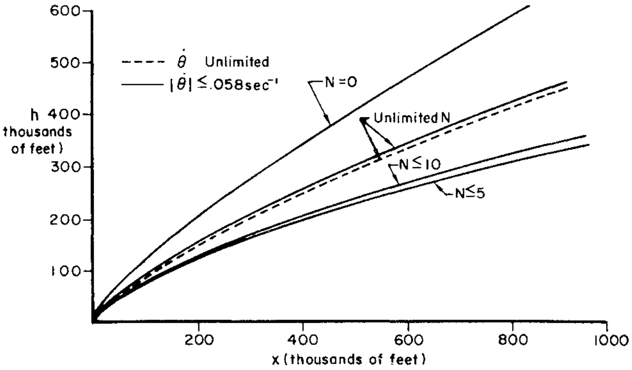

Fig. 8 Ascent profiles of maximum-range trajectories.

be obtained from the general program. In the first, $W(t)$ can be chosen arbitrarily large after booster burnout. An original choice of $\alpha = 0$ during re-entry will then remain unchanged. In the second case, $|\dot{\theta}_{max}|$ and N_{max} can be taken as arbitrarily large. In this case α will drop arbitrarily quickly from zero at $t = 3$ to its optimal value at that time.

The first case will be familiar to a host of vehicle ascent optimizers. Usually, the impact point is computed from the vacuum ellipse equation so that the numerical integration need be carried only to booster burnout. In the program here, the integration is all the way to impact in order to include the skipping possibility. Integration to impact has no important effect on the nature of the optimal ascent when re-entry is ballistic. Now, the tradeoff between flight-path angle and velocity at burnout gives a very wide band of burnout conditions that produce nearly maximum range. Since our optimization technique obtains improvements proportional to the gradient, very little can be accomplished after the nearly maximum range is achieved. The technique converges rapidly to approximately optimal payoff but then converges slowly toward the exact extremal. This is fully satisfactory when a physical understanding of the problem is present, but it can be worrisome in the opposite circumstance.

The most interesting results are shown in Figs. 8–11 which give the trajectories and the $\alpha(t)$ programs obtained. It was found inadvisable to plot complete histories on a single scale because of the different natures of the boost and re-entry phases.

Figure 8 consists of ascent trajectories, with altitude and range scales the same. They are seen to be smooth, the essential differences being in tilt programs and the resulting burnout conditions. Figure 9 gives $\alpha(t)$ programs during boost. It should be observed that there is a considerable variety of $\alpha(t)$ programs which will give essentially the same burnout conditions. This is because the "averaged" effect of the $\alpha(t)$ program produces the burnout conditions. The first

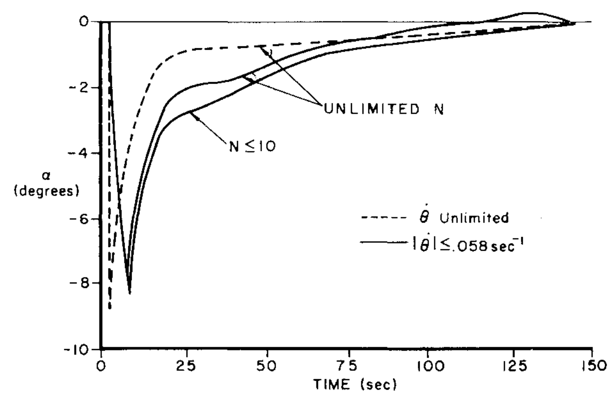

Fig. 9 Angle-of-attack programs for the ascent phase of maximum-range trajectories.

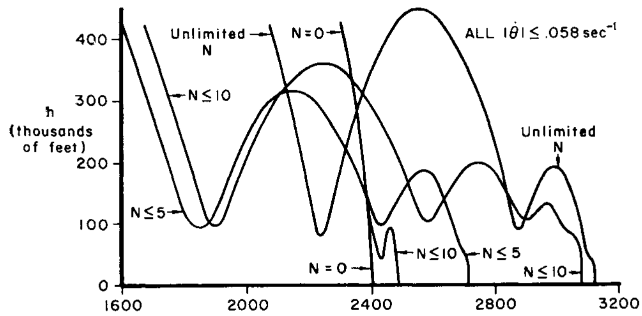

Fig. 10 Re-entry profiles of maximum-range trajectories.

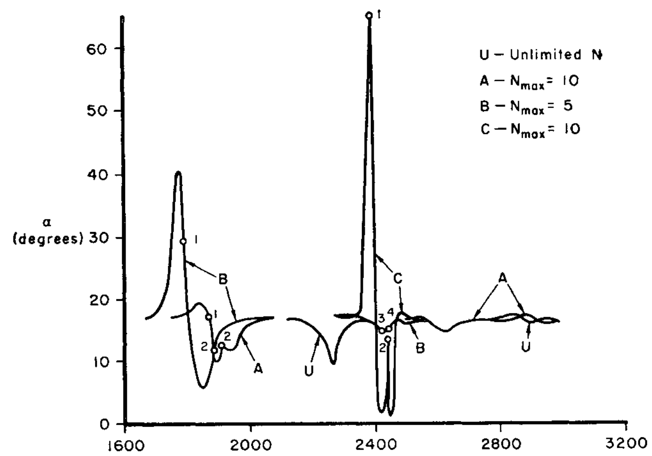

Fig. 11 Angle-of-attack programs for the re-entry phase of maximum-range trajectories.

few seconds are most influential because of the low velocity. A small shift in the initial tip phase, in the order of 0.1° for 10 sec, may be offset by a shift of 0.1° or 0.2° in the opposite direction over the rest of the ascent. Thus, there is no real significance to the detailed shape or position of the $\alpha(t)$ curves for $t > 20$ sec. Roughly speaking, the missile tips over to approximately the "correct" flight-path angle and then flies close to a gravity turn over most of the ascent.

Three $\alpha(t)$ programs are shown in Fig. 9. The program with unlimited pitch rate (and unlimited normal load) drops nearly to $-9°$ at the 3-sec point and then rises to $-1°$ at 20 sec. The limited pitch-rate program (still with unlimited normal load) descends over 5 sec and then rises more slowly than the constrained case. Since vehicle pitch produces greater turning of the velocity vector with smaller velocity, the average negative α over the initial tip period need not be so great when the pitch rate is unconstrained. This is seen to be the case in Fig. 9 and in the resultant trajectories in Fig. 8. The ascent trajectories for the two cases discussed are seen to be very close; a little is gained by tipping faster and doing less work against gravity.

The trajectories with normal load constrained turn down more during boost so as to achieve a shallower re-entry [except for one ($N \leq 10$) case to be discussed]. The α program for the (longer) $N \leq 10$ trajectory is given in Fig. 9. It is seen to maintain maximum pitch rate for about 1 sec more than the unconstrained normal force case and to have a more negative thrust-direction angle throughout. This produces a lower flight-path angle. The α program for the $N \leq 5$ trajectory is not shown. From the comparison of the N_{max} of 10 and 5 trajectories in Fig. 8, it is clear that it pulls down just a little more than the previous case.

In all cases the convergence to time t_1 (coming off the maximum-pitch-rate curve) was trouble-free. On the first iteration, if the nominal was moderately far from optimal, t_1 would change by several tenths of a second. After a half dozen iterations, the change would be 0.01 or 0.02 sec per improvement.

The re-entry phase is considerably more interesting in the comparisons that may be drawn. In particular, the $\alpha(t)$ programs vary markedly among the cases. The convergence of these maximum-range trajectories has one major feature in common: the importance of the weighting function $W(t)$. In this problem it is clear that a given $\delta\alpha$ in the boost phase is dramatically more important than in the re-entry phase, because the thrust direction, as well as the aerodynamic forces, is determined by α. This makes Λ orders of magnitude greater during boost than re-entry. If W = const were used, essentially all the $\delta\alpha$ would occur during boost ($\delta\alpha$ is proportional to Λ), with the result, after a few iterations, that the α program would oscillate from iteration to iteration during boost and remain unchanged during re-entry. To prevent this, a simple $W(t)$ choice was made: namely, one constant during boost and a different constant during re-entry. It was found typically that a ratio of roughly 100 worked efficiently in obtaining trajectories within 2% of maximum range. After that, the influence of boost on re-entry tended to stall progress. The difficulty lay in having relatively small changes in the boost period change the time of re-entry sufficiently to throw the re-entry $\alpha(t)$ program out of phase. This tended to invalidate the superposition assumption. [Small $\delta\alpha(t_a)$ is assumed not to affect appreciably the influence of small $\delta\alpha(t_b)$, $t_b \neq t_a$.] The most promising recourse at this point was to choose $W(t)$ so as to have no $\delta\alpha$ during boost and complete the re-entry optimization. After two or three re-entry improvements alone, a small change in the boost could be made. After three or four such cycles, the surface ranges were within 2 miles of ultimate maximum. That this is successful is due to the small sensitivity of total range to small changes in burnout conditions. In some problems, the last 1 or 2% might be so difficult to achieve by steepest-ascent as to be not worthwhile. In this situation an iterative method that requires an approximate extremal to begin might give rapid completion to the exact optimum.

In considering the re-entry trajectories, shown in Fig. 10 we might first notice the ballistic re-entry curve. The total range is 2420 naut miles, shortest of the group. (All curves in Figs. 10 and 11 were pitch-rate limited in ascent.) The burnout flight-path angle was greatest, 31°, and the range at re-entry (arbitrarily defined to occur at 320,000 ft) was greatest of the group. In Fig. 10 is shown a trajectory entering with the same conditions as the ballistic re-entry. This is an $N_{max} = 10$ trajectory, range maximized. "Maximized," of course, means a local maximum, since the optimization procedure employs a local gradient. This trajectory simply adds as much range with $N \leq N_{max}$ over the ballistic re-entry as can be achieved through re-entry skip alone. The solution is stationary. The final range is 2480 naut miles, an increase of only 60 miles over the $N = 0$ case. This $N_{max} = 10$ ascent trajectory is the same as the $N = 0$ ascent trajectory shown in Fig. 8.

Also in Fig. 10 is shown an $N_{max} = 10$ trajectory re-entering at 1740-naut-mile range, compared to 2340 for the last-discussed case. Although 600 naut miles are sacrificed at this point, the final range of 3070 is 590 naut miles greater. This is another completely different stationary solution with the same constraint. The additional range through skipping, made possible through a relatively shallow re-entry, is twice the "ballistic" range lost. In this particular case the nominal trajectory selected for the $N_{max} = 10$ case led to the first result reported, the shorter range. Physical reasoning suggested that a stationary solution with a long skip should exist, and so a nominal with much lower burnout angle, and thus much shallower re-entry, was used in obtaining the greater maximum. It would be difficult to overemphasize the possibility of multiple stationary solutions in a general nonlinear problem.

The author presently sees no guaranteed-satisfaction approach but feels that the researcher who understands his problem has the odds greatly in his favor.

A plausibility argument can be constructed for the multiple solutions previously mentioned, based on the unconstrained-N case. It has a burnout angle of 24°, roughly midway between the $N_{max} = 10$ cases. Its re-entry angle is likewise midway between. Since N is not limited, it is able to pull up much more sharply during re-entry than the other cases. In the first case the tradeoff was in the direction of a longer exo-atmospheric glide to offset the limited skip. In the second case a considerable amount was given up prior to re-entry to get the multiple skip. The ultimate range in this case, however, is less than 40 naut miles shorter than the unlimited N trajectory. That maximum range is 3110 naut miles, made with just two skips, whereas the 3070-naut-mile $N_{max} = 10$ trajectory requires three skips to get its range. In the $N_{max} = 10$ cases it appears that the tradeoff between ballistic and skip ranges is in different directions, depending on whether or not the nominal trajectory used is steeper than the unconstrained $N_{optimal}$. The author feels that only one stationary solution exists for the unconstrained extremal, and that it could be obtained starting with either of the $N_{max} = 10$ stationary solutions.

The $N_{max} = 5$ trajectory is different from the longer $N_{max} = 10$ trajectory only in degree. (It is felt that a second stationary solution exists, just as for $N_{max} = 10$.) Slightly lower burnout, and thus shorter re-entry range, is used, and enough energy is lost in the first pull-up that only two skips are achieved, and the resultant range is 2710 naut miles. This represents a loss of 400 naut miles over the unconstrained-N case, which has a peak N of approximately 30. A limit $N_{max} = 10$ causes very little loss, but dropping N_{max} further to 5 cuts more than half the range available from skip.

Finally, for this example, let us examine the re-entry $\alpha(t)$ programs of Fig. 11. In each case the nominal re-entry $\alpha(t)$ was chosen as a constant 17°. This value of α then appears unchanged for those portions of the trajectories above 200,000 ft. The α programs are plotted vs surface range rather than time so that the comparisons with the trajectories may be made.

The unconstrained-N case has α decreased during the major "bounce." In this case decreasing the drag was more beneficial than increasing the time to make the skip-out was harmful. The second skip is much less important; slightly larger (than 17°) α occurs during the pull-up and slightly smaller in the skipping out. If the optimization were pushed to the limit, substantially greater changes during the second bounce might occur, but they would yield only one or two miles additional range.

In each of the other trajectories, there is some region of $N = N_{max}$. The beginnings of these intervals are marked in Fig. 9 by the number 1 adjacent to small circles on the curves. The ends of the intervals are similarly marked by the number 2. The $N_{max} = 5$ trajectory has a short second interval of $N = 5$, in the second skip. The ends of that interval are marked by circles with numbers 3 and 4.

Again note that the α programs during second or third skip are not to be considered as carefully optimized. The potential additional gain possible in these regions is, at any event, negligible. The author feels, however, that the α programs in the first bounce are very good approximations to the exact extremals, as they have been obtained with what are generally unnecessarily long sequences of improvements. The computer program used here can obtain near maximum ranges (within 20 naut miles) after a sequence of 10 improvements. To gain additional assurance, approximately 20 to 30 improvements were made on the more difficult cases here.

As a final note, one should recognize the crucial role of $W(t)$ in determining the rate of convergence. In this problem the requirement for a large variation of W was apparent. In nearly every problem, a knowledgable analyst can choose W to suit his problem and gain anywhere from a minor savings in runs required, to a close approximation of the desired extremal rather than a futile series of oscillating control programs.

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